Math 142 Lecture 4 Notes

Daniel Raban

January 25, 2018

1 Identification Spaces and Attaching Maps

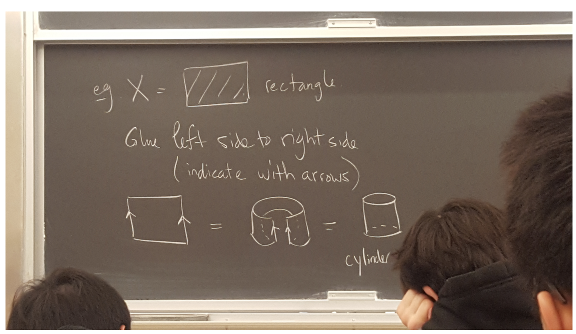
1.1 Identification spaces

How do we construct new topological spaces? We have already covered

1. subspaces
2. disjoint unions
3. product spaces[1].

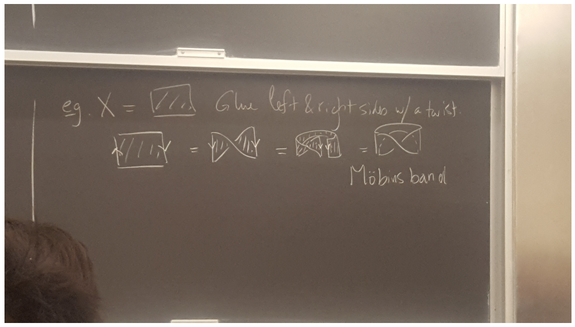
We will add identification spaces to the list. The ideas is that we start with a topological space X and "identify"/"set equal"/"glue" some subsets.

Example 1.1. Let X be a rectangle, and glue the left side to the right side. We indicate the gluing with arrows. Here, we get a cylinder.

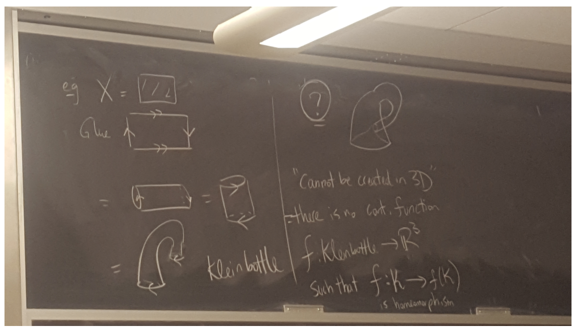


Example 1.2. Let X be a rectangle, and glue the left and right sides, but with a twist. We indicate this with the arrows on our diagram. We get a Möbius band.

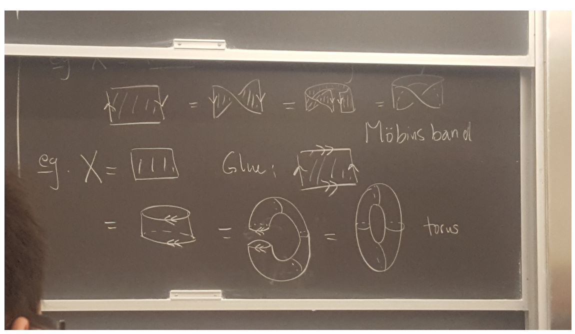
[1] What we called the "product topology" is actually the *box topology*, but these two coincide for products of finitely many spaces.



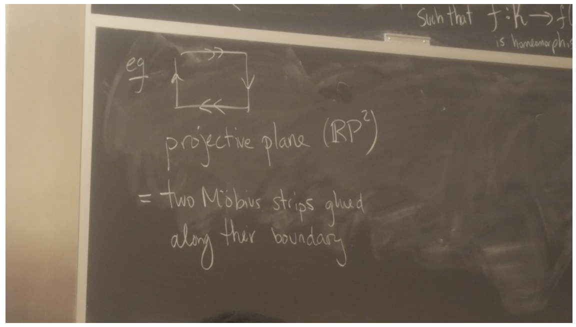
Example 1.3. Let X be a rectangle, and glue the left and right sides with no twists. The, glue the top and bottom together with no twists. We get a torus.



Example 1.4. Let X be a rectangle, and glue the top and bottom the same way, but glue the left and right sides together with a twist. We get a Klein bottle, but this "cannot be created in 3D." More precisely, there is no continuous function f : Klein bottle $\to \mathbb{R}^3$ such that $f : K \to f(K)$ is a homeomorphism.



Example 1.5. Let X be a rectangle, and glue the top and bottom with a twist and the left and right sides together with a twist. We get something called the "projective plane ($\mathbb{R}P^2$), which is two Möbius strips glued along their boundary. This also cannot be created in 3D.



Let's give a more formal definition.

Definition 1.1. If X is a topological space, let a partition $\mathcal{P}$ be a collection of nonempty subsets of X such that each $x \in X$ is in exactly one subset $A_x \in \mathcal{P}$. Write $\pi : X \to P$ sending $x \mapsto A_x$. Then make a new space Y (the *identification space*), by setting the points of Y to be elements in P, and $A \subseteq Y$ is open iff $\pi^{-1}(A) \subseteq X$; i.e. π is actually a map $\pi : X \to Y$, and the topology on Y is the largest so that π is continuous. This is the *identification topology.*

Example 1.6. Look at the uniq square $[0, 1] \times [0, 1] \subseteq \mathbb{R}^2$. To make a cylinder, set P to include the subsets:

- one singleton subset $\{x\}$ for each $x \in (0, 1) \times [0, 1]$
- $\{(0, y), (1, y)\}$ for each $y \in [0, 1]$

Remark 1.1. In some of our other examples, we need to also put all four corners of the rectangle into one subset.

Theorem 1.1. *If Y is an identification space, and Z is any space, then $f : Y \to Z$ is continuous iff $f \circ \pi : X \to Z$ is continuous.*

1.2 Attaching maps

Definition 1.2. Let X, Y be topological spaces, $A \subseteq X$ be a subspace, and $f : X \to Y$ be a continuous map. Start with $X \amalg Y$, and let $\mathcal{P}$ have the subsets

- $f^{-1}(y) \cup \{y\}$ for $y \in f(A)$
- $\{x\}$ for each $x \in X \setminus A$
- $\{y\}$ for each $y \in T \setminus f(A)$.

We call the identification space $X \cup_f Y$; here f is called the *attaching map.*

Here is a special example of this construction.

Definition 1.3. Let X be any space, $A \subseteq X$, $Y = \{*\}$ (a space containing only 1 point), and $f : A \to Y$ be $a \mapsto *$.So $\mathcal{P}$ has

- $\{x\}$ for $x \in X \setminus A$
- $A \cup \{*\}$.

The identification space $X \cup_f Y$ is called the *quotient space* X/A.

Here, we have crushed A to a point.

Example 1.7. Let X be an interval and A be the boundary (the two endpoints). Then X/A is the circle S^1.

Example 1.8. Let X be a disc and A be the boundary (a circle). Then X/A is the sphere S^2.

You might have more trouble believing this. Think of bending your disc into the shape of the sphere, missing a patch at the top. If we condense the boundary to a single point, this closes the sphere.

Example 1.9. Let $X = B^n$ be an n-dimensional ball and $A = S^{n-1}$ be its boundary. Then $X/A \cong S^n$.

Remark 1.2. While these pictures may help with intuition, they are not exactly precise. We are not actually bending anything in our construction; we are identifying points together.

Theorem 1.2. *If $f : X \to Y$ is continuous and surjective, and if f maps open sets to open sets (or closed sets to closed sets), then Y is an identification space, and f is the projection map (π).*

Proof. Define a partition $\mathcal{P}$ of X that has subsets $f^{-1}(y)$ for each $y \in Y$. Here, surjectivity implies that $f^{-1}(y) \neq \varnothing$ for every y. We want to show that the identification space from $\mathcal{P}$ is homeomorphic to Y; i.e. we want to show that the topology on Y is the larges so that f is continuous. In other words, we need to show that if $f^{-1}(A) \subseteq X$ is open, then $A \subseteq Y$ is open.

Suppose f takes open sets to open sets. Since f is surjective, $f(f^{-1}(A)) = A$. So if $f^{-1}(A)$ is open, then $A = f(f^{-1}(A))$ is open by hypothesis. The case of f sending closed sets to closed sets is similar, except it includes taking complements. $\square$

Next time, we will show that if

$$X = B^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\},$$

$$A = S^{n-1} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\},$$

then $B^n/S^{n-1} \cong S^n$.

Here's something to think about before next lecture: crushing S^{n-1} to a point is the same as gluing 2 copies of B^n together along their boundaries. Why?